Part IV
Infinite Series

The 5th Wave By Rich Tennant

"Infinite series has to do with sequences. The sequence I remember best is n+1 over n² equals dazed expression plus pounding headache."

In this part . . .

I introduce the infinite series — that is, the sum of an infinite number of terms. I show you the basics of working with sequences and series, and show you a bunch of ways to determine whether a series is convergent or divergent. You also discover how to use the Taylor series for expressing and evaluating a wide variety of functions.

Chapter 11

Following a Sequence, Winning the Series

In This Chapter

- Knowing a variety of notations for sequences
- Telling whether a sequence is convergent or divergent
- Expressing series in both sigma notation and expanded notation
- Testing a series for convergence or divergence

Just when you think the semester is winding down, your Calculus II professor introduces a new topic: infinite series.

When you get right down to it, series aren't really all that difficult. After all, a series is just a bunch of numbers added together. Sure, it happens that this bunch is infinite, but addition is just about the easiest math on the planet.

But then again, the last month of the semester is crunch time. You're already anticipating final exams and looking forward to a break from studying. By the time you discover that the prof isn't fooling and really does expect you to know this material, infinite series can lead you down an infinite spiral of despair: Why *this?* Why *now?* Why *me?*

In this chapter, I show you the basics of series. First, you wade into these new waters slowly by examining infinite sequences. When you understand sequences, series make a whole lot more sense. Next, I introduce you to infinite series. I discuss how to express a series in both expanded notation and sigma notation, and then I make sure that you're comfortable with sigma notation. I also show you how every series is related to two sequences.

Next, I introduce you to the all-important topic of convergence and divergence. This concept looms large, so I give you the basics in this chapter and save the more complex information for Chapter 12. Finally, I introduce you to a few important types of series.

Introducing Infinite Sequences

A sequence of numbers is simply a bunch of numbers in a particular order. For example:

$1, 4, 9, 16, 25, \ldots$

$\pi, 2\pi, 3\pi, 4\pi, \ldots$

$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

$2, 3, 5, 7, 11, 13, \ldots$

$2, -2, 2, -2, \ldots$

$0, 1, -1, 2\ -2, 3, \ldots$

When a sequence goes on forever, it's an *infinite sequence.* Calculus — which focuses on all things infinite — concerns itself predominantly with infinite sequences.

Each number in a sequence is called a *term* of that sequence. So, in the sequence 1, 4, 9, 16, ... , the *first term* is 1, the *second term* is 4, and so forth.

Understanding sequences is an important first step toward understanding series.

Understanding notations for sequences

The simplest notation for defining a sequence is a variable with the subscript *n* surrounded by braces. For example:

$\{a_n\} = \{1, 4, 9, 16, \ldots\}$

$\{b_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

$\{c_n\} = \{4\pi, 6\pi, 8\pi, 10\pi, \ldots\}$

You can reference a specific term in the sequence by using the subscript:

$a_1 = 1 \qquad b_3 = \frac{1}{3} \qquad c_6 = 14\pi$

WARNING!

Make sure that you understand the difference between notation with and without braces:

- The notation $\{a_n\}$ *with* braces refers to the *entire* sequence.
- The notation a_n *without* braces refers to the *n*th term of the sequence.

When defining a sequence, instead of listing the first few terms, you can state a rule based on n. (This is similar to how a function is typically defined.) For example:

$\{a_n\}$, where $a_n = n^2$

$\{b_n\}$, where $b_n = \frac{1}{n}$

$\{c_n\}$, where $c_n = 2(n + 1)\pi$

Sometimes, for increased clarity, the notation includes the first few terms plus a rule for finding the nth term of the sequence. For example:

$\{a_n\} = \{1, 4, 9, \ldots, n^2, \ldots\}$

$\{b_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$

$\{c_n\} = \{4\pi, 6\pi, 8\pi, \ldots, 2(n + 1)\pi, \ldots\}$

This notation can be made more concise by appending starting and ending values for n:

$$\{a_n\} = \{n^2\}_{n=1}^{\infty}$$

$$\{b_n\} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

$$\{c_n\} = \{2n\pi\}_{n=2}^{\infty}$$

This last example points out the fact that the initial value of n doesn't have to be 1, which gives you greater flexibility to define a number series by using a rule.

TIP

Don't let the fancy notation for number sequences get to you. When you're faced with a new sequence that's defined by a rule, jot down the first four or five numbers in that sequence. Usually, after you see the pattern, you'll find that a problem is much easier.

Looking at converging and diverging sequences

Every infinite sequence is either convergent or divergent:

- A *convergent* sequence has a limit — that is, it approaches a real number.
- A *divergent* sequence doesn't have a limit.

For example, here's a convergent sequence:

$$\{a_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...\}$$

This sequence approaches 0, so:

$$\lim\{a_n\} = 0$$

Thus, this sequence *converges to 0.*

Here's another convergent sequence:

$$\{b_n\} = \{7, 9, 7\frac{1}{2}, 8\frac{1}{2}, 7\frac{3}{4}, 8\frac{1}{4}, ...\}$$

This time, the sequence approaches 8 from above and below, so:

$$\lim\{b_n\} = 8$$

In many cases, however, a sequence *diverges* — that is, it fails to approach any real number. Divergence can happen in two ways. The most obvious type of divergence occurs when a sequence explodes to infinity or negative infinity — that is, it gets farther and farther away from 0 with every term. Here are a few examples:

$$-1, -2, -3, -4, -5, -6, -7, ...$$

$$\ln 1, \ln 2, \ln 3, \ln 4, \ln 5, ...$$

$$2, 3, 5, 7, 11, 13, 17, ...$$

In each of these cases, the sequence approaches either ∞ or $-\infty$, so the limit of the sequence *does not exist* (DNE). Therefore, the sequence is divergent.

A second type of divergence occurs when a sequence oscillates between two or more values. For example:

$$0, 7, 0, 7, 0, 7, 0, 7, ...$$

$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, ...$$

In these cases, the sequence bounces around indefinitely, never settling in on a value. Again, the limit of the sequence *does not exist,* so the sequence is divergent.

Introducing Infinite Series

In contrast to an infinite sequence (which is an endless list of numbers), an *infinite series* is an endless sum of numbers. You can change any infinite sequence to an infinite series simply by changing the commas to plus signs. For example:

$$1, 2, 3, 4, \ldots \qquad 1 + 2 + 3 + 4 + \ldots$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \qquad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

$$1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \ldots \qquad 1 + -1 + \frac{1}{2} + -\frac{1}{2} + \frac{1}{4} + -\frac{1}{4} + \ldots$$

The two principal notations for series are sigma notation and expanded notation. *Sigma notation* provides an explicit rule for generating the series (see Chapter 2 for the basics of sigma notation). *Expanded notation* gives enough of the first few terms of a series so that the pattern generating the series becomes clear.

For example, here are three series defined using both forms of notation:

$$\sum_{n=1}^{\infty} 2n = 2 + 4 + 6 + 8 + \ldots$$

$$\sum_{n=0}^{\infty} \frac{1}{4^n} = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots$$

$$\sum_{n=3}^{\infty} \frac{n}{e^n} = \frac{3}{e^3} + \frac{4}{e^4} + \frac{5}{e^5} + \ldots$$

As you can see, a series can start at any integer.

As with sequences (see "Introducing Infinite Sequences" earlier in this chapter), every series is either convergent or divergent:

- A *convergent* series evaluates to a real number.
- A *divergent* series doesn't evaluate to a real number.

To get clear on how evaluation of a series connects with convergence and divergence, I give you a few examples. To start out, consider this convergent series:

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

Notice that as you add this series from left to right, term by term, the running total is a sequence that approaches 2:

$$1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots$$

This sequence is called the *sequence of partial sums* for this series. I discuss sequences of partial sums in greater detail later in "Connecting a Series with Its Two Related Sequences."

REMEMBER

For now, please remember that the value of a series equals the limit of its sequence of partial sums. In this case, because the limit of the sequence is 2, you can evaluate the series as follows:

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 2$$

Thus, this series *converges to 2.*

Often, however, a series *diverges* — that is, it doesn't equal any real number. As with sequences, divergence can happen in two ways. The most obvious type of divergence occurs when a series explodes to infinity or negative infinity. For example:

$$\sum_{n=1}^{\infty} -n = -1 + -2 + -3 + -4 + \ldots$$

This time, watch what happens as you add the series term by term:

$$-1, -3, -6, -10, \ldots$$

Clearly, this sequence of partial sums diverges to negative infinity, so the series is divergent as well.

A second type of divergence occurs when a series alternates between positive and negative values in such a way that the series never approaches a value. For example:

$$\sum_{n=0}^{\infty} (-1)^n = 1 + -1 + 1 + -1 + \ldots$$

So, here's the related sequence of partial sums:

$$1, 0, 1, 0, \ldots$$

In this case, the sequence of partial sums alternates forever between 1 and 0, so it's divergent; therefore, the series is also divergent. This type of series is called, not surprisingly, an *alternating series.* I discuss alternating series in greater depth in Chapter 12.

Convergence and divergence are arguably the most important topics in your final weeks of Calculus II. Many of your exam questions will ask you to determine whether a given series is convergent or divergent.

Later in this chapter, I show you how to decide whether certain important types of series are convergent or divergent. Chapter 12 gives you a ton of handy tools for answering this question more generally. For now, just keep this important idea of convergence and divergence in mind.

Getting Comfy with Sigma Notation

Sigma notation is a compact and handy way to represent series.

Okay — that's the official version of the story. What's also true is that sigma notation can be unclear and intimidating — especially when the professor starts scrawling it all over the blackboard at warp speed while explaining some complex proof. Lots of students get left in the chalk dust (or dry-erase marker fumes).

At the same time, sigma notation is useful and important because it provides a concise way to express series and mathematically manipulate them.

In this section, I give you a bunch of handy tips for working with sigma notation. Some of the uses for these tips become clearer as you continue to study series later in this chapter and in Chapters 12 and 13. For now, just add these tools to your toolbox and use them as needed.

Writing sigma notation in expanded form

When you're working with an unfamiliar series, begin by writing it out using both sigma and expanded notation. This practice is virtually guaranteed to increase your understanding of the series. For example:

$$\sum_{n=1}^{\infty} \frac{2^n}{3n}$$

As it stands, you may not have much insight into what this series looks like, so expand it out:

$$\sum_{n=1}^{\infty} \frac{2^n}{3n} = \frac{2}{3} + \frac{4}{6} + \frac{8}{9} + \frac{16}{12} + \frac{32}{16} + \dots$$

As you spend a bit of time generating this series, it begins to grow less frightening. For one thing, you may notice that in a race between the numerator and denominator, eventually the numerator catches up and pulls ahead. Because the terms eventually grow greater than 1, the series explodes to infinity, so it *diverges.*

Seeing more than one way to use sigma notation

Virtually any series expressed in sigma notation can be rewritten in a slightly altered form. For example:

$$\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \ldots$$

You can express this series in sigma notation as follows:

$$\sum_{n=3}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \ldots$$

Alternatively, you can express the same series in any of the following ways:

$$= \sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^{n+1}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n+2}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+3}$$

TECHNICAL STUFF

Depending on the problem that you're trying to solve, you may find one of these expressions more advantageous than the others — for example, when using the comparison tests that I introduce in Chapter 12. For now, just be sure to keep in mind the flexibility at your disposal when expressing a series in sigma notation.

Discovering the Constant Multiple Rule for series

In Chapter 4, you discover that the Constant Multiple Rule for Integration allows you to simplify an integral by factoring out a constant. This option is also available when you're working with series. Here's the rule:

$$\Sigma\, ca_n = c\, \Sigma\, a_n$$

For example:

$$\sum_{n=1}^{\infty} \frac{7}{n^2} = 7 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

To see why this rule works, first expand the series so that you can see what you're working with:

$$\sum_{n=1}^{\infty} \frac{7}{n^2} = 7 + \frac{7}{4} + \frac{7}{9} + \frac{7}{16} + \ldots$$

Working with the expanded form, you can factor out a 7 from each term:

$$= 7\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots\right)$$

Now, express the contents of the parentheses in sigma notation:

$$= 7 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

As if by magic, this procedure demonstrates that the two sigma expressions are equal. But, this magic is really nothing more exotic than your old friend from grade school, the distributive property.

Examining the Sum Rule for series

Here's another handy tool for your growing toolbox of sigma tricks. This rule mirrors the Sum Rule for Integration (see Chapter 4), which allows you to split a sum inside an integral into the sum of two separate integrals. Similarly, you can break a sum inside a series into the sum of two separate series:

$$\Sigma\,(a_n + b_n) = \Sigma\, a_n + \Sigma\, b_n$$

For example:

$$= \sum_{n=1}^{\infty} \frac{n+1}{2^n}$$

A little algebra allows you to split this fraction into two terms:

$$= \sum_{n=1}^{\infty} \left(\frac{n}{2^n} + \frac{1}{2^n}\right)$$

Now, the rule allows you to split this result into two series:

$$= \sum_{n=1}^{\infty} \frac{n}{2^n} + \sum_{n=1}^{\infty} \frac{1}{2^n}$$

This sum of two series is equivalent to the series that you started with. As with the Sum Rule for Integration, expressing a series as a sum of two simpler series tends to make problem-solving easier. Generally speaking, as you proceed onward with series, any trick you can find to simplify a difficult series is a good thing.

Connecting a Series with Its Two Related Sequences

Every series has two related sequences. The distinction between a sequence and a series is as follows:

- A sequence is a *list* of numbers separated by *commas* (for example: 1, 2, 3, ...).
- A series is a *sum* of numbers separated by *plus signs* (for example: 1 + 2 + 3 + ...).

When you see how a series and its two related sequences are distinct but also related, you gain a clearer understanding of how series work.

A series and its defining sequence

The first sequence related to a series is simply the sequence that defines the series in the first place. For example, here are three series written in both sigma notation and expanded notation, each paired with its defining sequence:

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \qquad 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$$

$$\sum_{n=1}^{\infty}\frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \dots \qquad \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$$

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \qquad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

When a sequence $\{a_n\}$ is already defined, you can use the notation $\Sigma\, a_n$ to refer to the related series starting at $n = 1$. For example, when $\{a_n\} = \frac{1}{n^2}$, $\Sigma\, a_n = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$.

Understanding the distinction between a series and the sequence that defines it is important for two reasons. First, and most basic, you don't want to get the concepts of sequences and series confused. But second, the sequence that defines a series can provide important information about the series. See Chapter 12 to find out about the *n*th term test, which provides a connection between a series and its defining sequence.

A series and its sequences of partial sums

You can learn a lot about a series by finding the *partial sums* of its first few terms. For example, here's a series that you've seen before:

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

And here are the first four partial sums of this series:

$$\sum_{n=1}^{1}\left(\frac{1}{2}\right)^n = \frac{1}{2}$$

$$\sum_{n=1}^{2}\left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$\sum_{n=1}^{3}\left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$\sum_{n=1}^{4}\left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$

You can turn the partial sums for this series into a sequence as follows:

$$\{S_n\} = \left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots, \frac{2^n}{2^n - 1}, \ldots\right\}$$

In general, every series $\Sigma\, a_n$ has a related sequence of partial sums $\{S_n\}$. For example, here are a few such pairings:

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \qquad \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots$$

$$\sum_{n=1}^{\infty}\frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \ldots \qquad \frac{1}{2}, \frac{7}{6}, \frac{23}{12}, \frac{163}{60}, \ldots$$

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \ldots \qquad 1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \ldots$$

Every series and its related sequence of partial sums are either *both convergent* or *both divergent.* Moreover, if they're both convergent, both converge to the same number.

This rule should come as no big surprise. After all, a sequence of partial sums simply gives you a running total of where a series is going. Still, this rule can be helpful. For example, suppose that you want to know whether the following sequence is convergent or divergent:

$$1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \ldots$$

What the heck is this sequence, anyway? Upon deeper examination, however, you discover that it's the sequence of partial sums for every simple series:

$$1$$

$$1 + \frac{1}{2} = \frac{3}{2}$$

$$1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{137}{60}$$

This series, called the harmonic series, is divergent, so you can conclude that its sequence of partial sums also diverges.

Recognizing Geometric Series and P-Series

At first glance, many series look strange and unfamiliar. But a few big categories of series belong in the Hall of Fame. When you know how to identify these types of series, you have a big head start on discovering whether they're convergent or divergent. In some cases, you can also find out the exact value of a convergent series without spending all eternity adding numbers.

In this section, I show you how to recognize and work with two common types of series: geometric series and *p*-series.

Getting geometric series

A geometric series is any series of the following form:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \ldots$$

Here are a few examples of geometric series:

$$\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + 16 + \ldots$$

$$\sum_{n=0}^{\infty} \frac{1}{10^n} = 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1{,}000} + \ldots$$

$$\sum_{n=0}^{\infty} \frac{3}{100^n} = 3 + \frac{3}{100} + \frac{3}{10{,}000} + \frac{3}{1{,}000{,}000} + \ldots$$

In the first series, $a = 1$ and $r = 2$. In the second, $a = 1$ and $r = \frac{1}{10}$. And in the third, $a = 3$ and $r = \frac{1}{100}$.

If you're unsure whether a series is geometric, you can test it as follows:

1. **Let a equal the first term of the series.**
2. **Let r equal the second term divided by the first term.**
3. **Check to see whether the series fits the form $a + ar^2 + ar^3 + ar^4 + \ldots$.**

For example, suppose that you want to find out whether the following series is geometric:

$$\frac{8}{5} + \frac{6}{5} + \frac{9}{10} + \frac{27}{40} + \frac{81}{160} + \frac{243}{640} + \ldots$$

Use the procedure I outline as follows:

1. **Let a equal the first term of the series:**

   $$a = \frac{8}{5}$$

2. **Let r equal the second term divided by the first term:**

   $$r = \frac{6}{5} \div \frac{8}{5} = \frac{3}{4}$$

3. Check to see whether the series fits the form $a + ar^2 + ar^3 + ar^4 + \ldots$:

$$a = \frac{8}{5}$$

$$ar = \frac{8}{5}\left(\frac{3}{4}\right) = \frac{6}{5}$$

$$ar^2 = \frac{6}{5}\left(\frac{3}{4}\right)^2 = \frac{9}{10}$$

$$ar^3 = \frac{9}{10}\left(\frac{3}{4}\right)^3 = \frac{27}{40}$$

As you can see, this series is geometric. To find the limit of a geometric series $a + ar + ar^2 + ar^3 + \ldots$, use the following formula:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

So, the limit of the series in the previous example is:

$$\sum_{n=0}^{\infty} \frac{8}{5}\left(\frac{3}{4}\right)^n = \frac{\frac{8}{5}}{1-\frac{3}{4}} = \frac{8}{5} \cdot \frac{1}{4} = \frac{32}{5}$$

When the limit of a series exists, as in this example, the series is called *convergent.* So, you say that this series *converges* to $\frac{32}{5}$.

In some cases, however, the limit of a geometric series does not exist (DNE). In that case, the series is *divergent.* Here's the complete rule that tells you whether a series is convergent or divergent:

For any geometric series $a + ar + ar^2 + ar^3 + \ldots$, if r falls in the open set $(-1, 1)$, the series converges to $\frac{a}{1-r}$; otherwise, the series diverges.

An example makes clear why this is so. Look at the following geometric series:

$$1 + \frac{5}{4} + \frac{25}{16} + \frac{125}{64} + \frac{625}{256} + \ldots$$

In this case, $a = 1$ and $r = \frac{5}{4}$. Because $r > 1$, each term in the series is greater than the term that precedes it, so the series grows at an ever-accelerating rate.

This series illustrates a simple but important rule of thumb for deciding whether a series is convergent or divergent: A series can be convergent only when its related sequence *converges to zero.* I discuss this important idea (called the *nth-term test*) further in Chapter 12.

Similarly, look at this example:

$$1 + -\frac{5}{4} + \frac{25}{16} + -\frac{125}{64} + \frac{625}{256} + \ldots$$

This time, $a = 1$ and $r = -\frac{5}{4}$. Because $r < -1$, the odd terms grow increasingly positive and the even terms grow increasingly negative. So the related sequence of partial sums alternates wildly from the positive to the negative, with each term further from zero than the preceding term.

A series in which alternating terms are positive and negative is called an *alternating series.* I discuss alternating series in greater detail in Chapter 12.

TIP

Generally speaking, the geometric series is the only type of series that has a simple formula to calculate its value. So, when a problem asks for the value of a series, try to put it in the form of a geometric series.

For example, suppose that you're asked to calculate the value of this series:

$$\frac{5}{7} + \frac{10}{21} + \frac{20}{63} + \frac{40}{189} + \ldots$$

The fact that you're being asked to calculate the value of the series should tip you off that it's geometric. Use the procedure I outline earlier to find a and r:

$$a = \frac{5}{7}$$

$$r = \frac{10}{21} \div \frac{5}{7} = \frac{2}{3}$$

So here's how to express the series in sigma notation as a geometric series in terms of a and r:

$$\sum_{n=1}^{\infty} \frac{5}{7}\left(\frac{2}{3}\right)^n = \frac{5}{7} + \frac{10}{21} + \frac{20}{63} + \frac{40}{189} + \ldots$$

At this point, you can use the formula for calculating the value of this series:

$$= \frac{a}{1-r} = \frac{\frac{5}{7}}{\left(1 - \frac{2}{3}\right)} = \frac{5}{7} \cdot \frac{1}{3} = \frac{15}{7}$$

Pinpointing p-series

Another important type of series is called the *p*-series. A *p-series* is any series in the following form:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$$

Here's a common example of a *p-series,* when $p = 2$:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$$

Here are a few other examples of *p*-series:

$$\sum_{n=1}^{\infty} \frac{1}{n^5} = 1 + \frac{1}{32} + \frac{1}{243} + \frac{1}{1{,}024} + \ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{2} + \frac{1}{\sqrt{5}} + \ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{-1}} = 1 + 2 + 3 + 4 + \ldots$$

Don't confuse *p*-series with geometric series (which I introduce in the previous section). Here's the difference:

- A geometric series has the variable *n* in the exponent — for example, $\Sigma \left(\frac{1}{2}\right)^n$.
- A *p*-series has the variable in the base — for example $\Sigma \frac{1}{n^2}$.

As with geometric series, a simple rule exists for determining whether a *p*-series is convergent or divergent.

A *p*-series converges when $p > 1$ and diverges when $p \le 1$.

I give you a proof of this theorem in Chapter 12. In this section, I show you why a few important examples of *p*-series are either convergent or divergent.

Harmonizing with the harmonic series

When $p = 1$, the *p*-series takes the following form:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

This *p*-series is important enough to have its own name: the *harmonic series.* The harmonic series is *divergent.*

Testing p-series when p = 2, p = 3, and p = 4

Here are the *p*-series when *p* equals the first few counting numbers greater than 1:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \ldots$$

Because $p > 1$, these series are all *convergent.*

Testing p-series when $p = \frac{1}{2}$

When $p = \frac{1}{2}$, the *p*-series looks like this:

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{2} + \frac{1}{\sqrt{5}} + \ldots$$

Because $p \le 1$, this series *diverges.* To see why it diverges, notice that when n is a square number, the nth term equals $\frac{1}{n}$. So this *p*-series includes every term in the harmonic series plus many more terms. Because the harmonic series is divergent, this series is also divergent.

Chapter 12

Where Is This Going? Testing for Convergence and Divergence

In This Chapter

- Understanding convergence and divergence
- Using the *n*th-term test to prove that a series diverges
- Applying the versatile integral test, ratio test, and root test
- Distinguishing absolute convergence and conditional convergence

Testing for convergence and divergence is The Main Event in your Calculus II study of series. Recall from Chapter 11 that when a series *converges,* it can be evaluated as a real number. However, when a series *diverges,* it can't be evaluated as a real number, because it either explodes to positive or negative infinity or fails to settle in on a single value.

In Chapter 11, I give you two tests for determining whether specific types of series (geometric series and *p*-series) are convergent or divergent. In this chapter, I give you seven more tests that apply to a much wider range of series.

The first of these is the *n*th-term test, which is sort of a no-brainer. With this under your belt, I move on to two comparison tests: the direct comparison test and the limit comparison test. These tests are what I call one-way tests; they provide an answer only if the series passes the test but not if the series fails it. Next, I introduce three two-way tests, which provide one answer if the series passes the test and the opposite answer if the series fails it. These tests are the integral test, the ratio test, and the root test.

Finally, I introduce you to alternating series, in which terms are alternately positive and negative. I contrast alternating series with positive series, which are the series that you're already familiar with, and I show you how to turn a positive series into an alternating series and vice versa. Then I show you how to prove whether an alternating series is convergent or divergent by using the alternating series test. To finish up, I introduce you to the important concepts of absolute convergence and conditional convergence.

Starting at the Beginning

When testing for convergence or divergence, don't get too hung up on where the series starts. For example:

$$\sum_{n=1,001}^{\infty} \frac{1}{n}$$

This is just a harmonic series with the first 1,000 terms lopped off:

$$= \frac{1}{1{,}001} + \frac{1}{1{,}002} + \frac{1}{1{,}003} + \ldots$$

These fractions may look tiny, but the harmonic series diverges (see Chapter 11), and removing a finite number of terms from the beginning of this series doesn't change this fact.

The lesson here is that, when you're testing for convergence or divergence, what's going on at the beginning of the series is irrelevant. Feel free to lop off the first few billion or so terms of a series if it helps you to prove that the series is convergent or divergent.

Similarly, in most cases you can add on a few terms to a series without changing whether it converges or diverges. For example:

$$\sum_{n=1,000}^{\infty} \frac{1}{n-1}$$

You can start this series anywhere from $n = 2$ to $n = 999$ without changing the fact that it diverges (because it's a harmonic series). Just be careful, because if you try to start the series from $n = 1$, you're adding the term $\frac{1}{0}$, which is a big no-no. However, in most cases you can extend an infinite series without causing problems or changing the convergence or divergence of the series.

Although eliminating terms from the beginning of a series doesn't affect whether the series is convergent or divergent, it *does* affect the sum of a convergent series. For example:

$$\left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

Lopping off the first few terms of this series — say, 1, $\frac{1}{2}$, and $\frac{1}{4}$ — doesn't change the fact that it's convergent. But it does change the value that the series converges to. For example:

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1$$

Using the nth-Term Test for Divergence

The *n*th-term test for divergence is the first test that you need to know. It's easy and it enables you to identify lots of series as divergent.

If the limit of sequence $\{a_n\}$ doesn't equal 0, then the series $\Sigma\, a_n$ is *divergent.*

To show you why this test works, I define a sequence that meets the necessary condition — that is, a sequence that doesn't approach 0:

$$\{a_n\} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1} \cdots$$

Notice that the limit of the sequence is 1 rather than 0. So, here's the related series:

$$\sum_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots$$

Because this series is the sum of an infinite number of terms that are very close to 1, it naturally produces an infinite sum, so it's divergent.

The fact that the limit of a sequence $\{a_n\}$ equals 0 doesn't necessarily imply that the series $\Sigma\, a_n$ is convergent.

For example, the harmonic sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ approaches 0, but (as I demonstrate in Chapter 11) the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \ldots$ is divergent.

When testing for convergence or divergence, always perform the *n*th-term test first. It's a simple test, and plenty of teachers test for it on exams because it's easy to grade but still catches the unwary student. ***Remember:*** If the defining sequence of a series doesn't approach 0, the series diverges; otherwise, you need to move on to other tests.

Let Me Count the Ways

Tests for convergence or divergence tend to fall into two categories: one-way tests and two-way tests.

One-way tests

A one-way test allows you to draw a conclusion only when a series passes the test, but not when it fails. Typically, *passing the test* means that a given condition has been met.

The *n*th-term test is a perfect example of a one-way test: If a series *passes* the test — that is, if the limit of its defining sequence doesn't equal 0 — the series is *divergent.* But if the series *fails* the test, you can draw no conclusion.

Later in this chapter, you discover two more one-way tests: the direct comparison test and the limit comparison test.

Two-way tests

A two-way test allows you to draw one conclusion when a series passes the test and the opposite conclusion when a series fails the test. As with a one-way test, *passing the test* means that a given condition has been met. *Failing the test* means that the negation of that condition has been met.

For example, the test for geometric series is a two-way test (see Chapter 11 to find out more about testing geometric series for convergence and divergence). If a series passes the test — that is, if r falls in the open set $(-1, 1)$ — then the series is convergent. And if the series fails the test — that is, if $r \leq -1$ or $r \geq 1$ — then the series is divergent.

Similarly, the test for *p*-series is also a two-way test (see Chapter 11 for more on this test).

Keep in mind that no test — even a two-way test — is *guaranteed* to give you an answer. Think of each test as a tool. If you run into trouble trying to cut a piece of wood with a hammer, it's not the hammer's fault: You just chose the wrong tool for the job.

Similarly, if you can't find a clever way to demonstrate either the condition or its negation required by a specific test, you're out of luck. In that case, you may need to use a different test that's better suited to the problem.

Later in this chapter, I show you three more two-way tests: the integral test, the ratio test, and the root test.

Using Comparison Tests

Comparison tests allow you to use stuff that you know to find out stuff that you want to know. The stuff that you know is more eloquently called a *benchmark series* — a series whose convergence or divergence you've already proven. The stuff that you want to know is, of course, whether an unfamiliar series converges or diverges.

As with the nth-term test, comparison tests are one-way tests: When a series passes the test, you prove what you've set out to prove (that is, either convergence or divergence). But when a series fails the test, the result of a comparison test in inconclusive.

In this section, I show you two basic comparison tests: the direct comparison test and the limit comparison test.

Getting direct answers with the direct comparison test

You can use the direct comparison test to prove either convergence or divergence, depending on how you set up the test.

To prove that a series converges:

1. **Find a benchmark series that you know converges.**
2. **Show that each term of the series that you're testing is less than or equal to the corresponding term of the benchmark series.**

To prove that a series diverges:

1. **Find a benchmark series that you know diverges.**
2. **Show that each term of the series you're testing is greater than or equal to the corresponding term of the benchmark series.**

For example, suppose that you're asked to determine whether the following series converges or diverges:

$$\text{Benchmark series: } \sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \ldots$$

It's hard to tell just by looking at it whether this particular series is convergent or divergent. However, it looks a bit like a p-series with $p = 2$:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$$

You know that this p-series converges (see Chapter 11 if you're not sure why), so use it as your benchmark series. Now, your task is to show that every term in the series that you're testing is less than the corresponding term of the benchmark series:

First term: $\frac{1}{2} < 1$

Second term: $\frac{1}{5} < \frac{1}{4}$

Third term: $\frac{1}{10} < \frac{1}{9}$

This looks good, but to complete the proof formally, here's what you want to show:

nth term: $\frac{1}{n^2+1} \leq \frac{1}{n^2}$

To see that this statement is true, notice that the numerators are the same, but the denominator $(n^2 + 1)$ is greater than n^2. So, the function $\frac{1}{n^2+1}$ is less than $\frac{1}{n^2}$, which means that every term in the test series is less than the corresponding term in the convergent benchmark series. Therefore, both series are convergent.

As another example, suppose that you want to test the following series for convergence or divergence:

$$\sum_{n=1}^{\infty} \frac{3}{n} = 3 + \frac{3}{2} + 1 + \frac{3}{4} + \frac{3}{5} + \ldots$$

This time, the series reminds you of the trusty harmonic series, which you know is divergent:

Benchmark series: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$

Using the harmonic series as your benchmark, compare the two series term by term:

First term: $3 > 1$

Second term: $\frac{3}{2} > \frac{1}{2}$

Third term: $1 > \frac{1}{3}$

Again, you have reason to be hopeful, but to complete the proof formally, you want to show the following:

nth term: $\frac{3}{n} \geq \frac{1}{n}$

This time, notice that the denominators are the same, but the numerator 3 is greater than the numerator 1. So the function $\frac{3}{n}$ is greater than $\frac{1}{n}$.

Again, you've shown that every term in the test series is greater than the corresponding term in the divergent benchmark series, so both series are divergent.

As a third example, suppose that you're asked to show whether this series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots$$

In this case, multiplying out the denominators is a helpful first step:

$$= \sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 2}$$

Now, the series looks a little like a *p*-series with $p = 2$, so make this your benchmark series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$$

The benchmark series converges, so you want to show that every term of the test series is less than the corresponding term of the benchmark. This looks likely because:

First term: $\frac{1}{6} < 1$

Second term: $\frac{1}{12} < \frac{1}{4}$

Third term: $\frac{1}{20} < \frac{1}{9}$

However, to convince the professor, you want to show that *every* term of the test series is less than the corresponding term:

*n*th term: $\frac{1}{n^2 + 3n + 2} \leq \frac{1}{n^2}$

As with the first example in this section, the numerators are the same, but the denominator of the test series is greater than that of the benchmark series. So the test series is, indeed, less than the benchmark series, which means that the test series is also convergent.

Testing your limits with the limit comparison test

As with the direct comparison test, the limit comparison test works by choosing a benchmark series whose behavior you know and using it to provide information about a test series whose behavior you don't know.

Here's the limit comparison test: Given a test series $\Sigma\, a_n$ and a benchmark series $\Sigma\, b_n$, find the following limit:

$$\lim_{n \to \infty} \frac{a_n}{b_n}$$

If this limit evaluates as a positive number, then either both series converge or both diverge.

As with the direct comparison test, when the test succeeds, what you learn depends upon what you already know about the benchmark series. If the benchmark series converges, so does the test series. However, if the benchmark series diverges, so does the test series.

Remember, however, that this is a one-way test: If the test fails, you can draw no conclusion about the test series.

The limit comparison test is especially good for testing infinite series based on *rational expressions*. For example, suppose that you want to see whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n-5}{n^2+1}$$

When testing an infinite series based on a rational expression, choose a benchmark series that's proportionally similar — that is, whose numerator and denominator differ by the same number of degrees.

In this example, the numerator is a first-degree polynomial and the denominator is a second-degree polynomial (for more on polynomials, see Chapter 2). So the denominator is one degree greater than the numerator. Therefore, I choose a benchmark series that's proportionally similar — the trusty harmonic series:

$$\text{Benchmark series: } \sum_{n=1}^{\infty} \frac{1}{n}$$

Before you begin, take a moment to get clear on what you're testing, and jot it down. In this case, you know that the benchmark series diverges. So, if the test succeeds, you prove that the test series also diverges. (If it fails, however, you're back to square one because this is a one-way test.)

Now, set up the limit (by the way, it doesn't matter which series you put in the numerator and which in the denominator):

$$\lim_{n \to \infty} \frac{\frac{n-5}{n^2+1}}{\frac{1}{n}}$$

At this point, you just crunch the numbers:

$$\lim_{n\to\infty} \frac{(n-5)\,n}{n^2+1}$$

$$=\lim_{n\to\infty} \frac{n^2-5n}{n^2+1}$$

Notice at this point that the numerator and denominator are both second-degree polynomials. Now, as you apply L'Hospital's Rule (taking the derivative of both the numerator and denominator), watch what happens:

$$=\lim_{n\to\infty} \frac{2n-5}{2n}$$

$$=\lim_{n\to\infty} \frac{2}{2} = 1$$

As if by magic, the limit evaluates to a positive number, so the test succeeds. Therefore, the test series diverges. Remember, however, that you made this magic happen by choosing a benchmark series in proportion to the test series.

Another example should make this crystal clear. Discover whether this series is convergent or divergent:

$$\lim_{n\to\infty} \frac{n^3-2}{4n^5-n^3-2}$$

When you see that this series is based on a rational expression, you immediately think of the limit comparison test. Because the denominator is two degrees higher than the numerator, choose a benchmark series with the same property:

$$\text{Benchmark series: } \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Before you begin, jot down the following: The benchmark converges, so if the test succeeds, the test series also converges. Next, set up your limit:

$$\lim_{n\to\infty} \frac{\dfrac{n^3-2}{4n^5-n^3-2}}{\dfrac{1}{n^2}}$$

Now, just solve the limit:

$$=\lim_{n\to\infty} \frac{(n^3-2)\,n^2}{4n^5-n^3-2}$$

$$=\lim_{n\to\infty} \frac{n^5-2n^2}{4n^5-n^3-2}$$

Again, the numerator and denominator have the same degree, so you're on the right track. Now, solving the limit is just a matter of grinding through a few iterations of L'Hospital's Rule:

$$\lim_{n \to \infty} \frac{5n^4 - 4n}{20n^4 - 3n^2}$$

$$= \lim_{n \to \infty} \frac{20n^3 - 4}{80n^3 - 6n}$$

$$= \lim_{n = \infty} \frac{60n^2}{240n^2 - 6}$$

$$= \lim_{n \to \infty} \frac{120n}{480n}$$

$$= \lim_{n \to \infty} \frac{120}{480} = \frac{1}{4}$$

The test succeeds, so the test series converges. And again, the success of the test was prearranged because you chose a benchmark series in proportion to the test series.

Two-Way Tests for Convergence and Divergence

Earlier in this chapter, I give you a variety of tests for convergence or divergence that work in one direction at a time. That is, passing the test gives you an answer, but failing it provides no information.

The tests in this section all have one important feature in common: Regardless of whether the series passes or fails, whenever the test gives you an answer, that answer *always* tells you whether the series is convergent or divergent.

Integrating a solution with the integral test

Just when you thought that you wouldn't have to think about integration again until two days before your final exam, here it is again. The good news is that the integral test gives you a two-way test for convergence or divergence.

Here's the integral test:

For any series of the form

$$\sum_{x=a}^{\infty} f(x)$$

consider its associated integral

$$\int_{a}^{\infty} f(x)\,dx$$

If this integral converges, the series also converges; however, if this integral diverges, the series also diverges.

In most cases, you use this test to find out whether a series converges or diverges by testing its associated integral. Of course, changing the series to an integral makes all the integration tricks that you already know and love available to you.

For example, here's how to use the integral test to show that the harmonic series is divergent. First, the series:

$$\sum_{x=1}^{\infty} \frac{1}{x} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

The integral test tells you that this series converges or diverges depending upon whether the following definite integral converges or diverges:

$$\int_{1}^{\infty} \frac{1}{x}\,dx$$

To evaluate this improper integral, express it as a limit, as I show you in Chapter 9:

$$= \lim_{c \to \infty} \int_{1}^{c} \frac{1}{x}\,dx$$

This is simple to integrate and evaluate:

$$= \lim_{c \to \infty} \left(\ln x \Big|_{x=1}^{x=c} \right)$$

$$= \lim_{c \to \infty} \ln c - \ln 1$$

$$\lim_{c \to \infty} \ln c - 0 = \infty$$

Because the limit explodes to infinity, the integral doesn't exist. Therefore, the integral test tells you that the harmonic series is divergent.

As another example, suppose that you want to discover whether the following series is convergent or divergent:

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Notice that this series starts at $n = 2$, because $n = 1$ would produce the term $\frac{1}{0}$. To use the integral test, transform the sum into this definite integral, using 2 as the lower limit of integration:

$$\int_{2}^{\infty} \frac{1}{x \ln x}\, dx$$

Again, rewrite this improper integral as the limit of an integral (see Chapter 9):

$$\lim_{c \to \infty} \int_{2}^{c} \frac{1}{x \ln x}\, dx$$

To solve the integral, use the following variable substitution:

$$u = \ln x$$

$$du = \frac{1}{x} dx$$

So you can rewrite the integral as follows:

$$\lim_{c \to \infty} \int_{\ln 2}^{\ln c} \frac{1}{u}\, du$$

Note that as the variable changes from x to u, the limits of integration change from 2 and c to ln 2 and ln c. This change arises when I plug the value $x = 2$ into the equation $u = \ln x$, so $u = \ln 2$. (For more on using variable substitution to evaluate definite integrals, see Chapter 5.)

At this point, you can evaluate the integral:

$$\lim_{c \to \infty} \left(\ln u \Big|_{u=\ln 2}^{u=\ln c} \right)$$

$$\lim_{c \to \infty} \ln(\ln c) - \ln(\ln 2) = \infty$$

You can see without much effort that as c approaches infinity, so does ln c, and the rest of the expression doesn't affect this. Therefore, the series that you're testing is divergent.

Rationally solving problems with the ratio test

The ratio test is especially good for handling series that include factorials. Recall that the factorial of a counting number, represented by the symbol !, is that number multiplied by every counting number less than itself. For example:

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

TIP

Flip to Chapter 2 for some handy tips on factorials that may help you in this section.

REMEMBER

To use the ratio test, take the limit (as n approaches ∞) of the $(n + 1)$th term divided by the nth term of the series:

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n + 1}$$

At the risk of destroying all the trust that you and I have built between us over these pages, I must confess that there are not two, but *three* possible outcomes to the ratio test:

- If this limit is less than 1, the series converges.
- If this limit is greater than 1, the series diverges.
- If this limit equals 1, the test is inconclusive.

But I'm sticking to my guns and calling this a two-way test, because — depending on the outcome — it can potentially prove either convergence or divergence.

For example, suppose that you want to find out whether the following series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

Before you begin, expand the series so that you can get an idea of what you're working with. I do this in two steps to make sure that the arithmetic is correct:

$$= \frac{2}{1} + \frac{2 \cdot 2}{2 \cdot 1} + \frac{2 \cdot 2 \cdot 2}{3 \cdot 2 \cdot 1} + \frac{2 \cdot 2 \cdot 2 \cdot 2}{4 \cdot 3 \cdot 2 \cdot 1} + \ldots$$

$$= 2 + 2 + \frac{4}{3} + \frac{2}{3} + \frac{4}{15} + \ldots$$

To find out whether this series converges or diverges, set up the following limit:

$$\lim_{n\to\infty}\frac{\dfrac{2^{n+1}}{(n+1)!}}{\dfrac{2^n}{n!}}$$

As you can see, I place the function that defines the series in the denominator. Then I rewrite this function, substituting $n + 1$ for n, and I place the result in the numerator. Now, evaluate the limit:

$$=\lim_{n\to\infty}\frac{(2^{n+1})(n!)}{(n+1)!(2^n)}$$

At this point, to see why the ratio test works so well for exponents and factorials, factor out a 2 from 2^{n+1} and an $n + 1$ from $(n + 1)!$:

$$=\lim_{n\to\infty}\frac{2(2^n)(n!)}{(n+1)(n!)(2^n)}$$

This trick allows you to simplify the limit greatly:

$$=\lim_{n\to\infty}\frac{2}{n+1}=0<1$$

Because the limit is less than 1, the series converges.

Rooting out answers with the root test

The root test works best with series that have powers of n in both the numerator and denominator.

REMEMBER

To use the root test, take the limit (as n approaches ∞) of the nth root of the nth term of the series:

$$\lim_{n\to\infty}\sqrt[n]{a_n}$$

As with the ratio test, even though I call this a two-way test, there are really three possible outcomes:

- If the limit is less than 1, the series converges.
- If the limit is greater than 1, the series diverges.
- If the limit equals 1, the test is inconclusive.

For example, suppose that you want to decide whether the following series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$$

This would be a very hairy problem to try to solve using the ratio test. To use the root test, take the limit of the nth root of the nth term:

$$\lim_{n \to \infty} \sqrt[n]{\frac{(\ln n)^n}{n^n}}$$

At first glance, this expression looks worse than what you started with. But it begins to look better when you separate the numerator and denominator into two roots:

$$= \lim_{n \to \infty} \frac{\sqrt[n]{(\ln n)^n}}{\sqrt[n]{n^n}}$$

Now, a lot of cancellation is possible:

$$= \lim_{n \to \infty} \frac{\sqrt[n]{\ln n}}{n}$$

Suddenly, the problem doesn't look so bad. The numerator and denominator both approach ∞, so apply L'Hospital's Rule:

$$= \lim_{n \to \infty} \frac{1}{n} = 0 < 1$$

Because the limit is less than 1, the series is convergent.

Alternating Series

Each of the series that I discuss earlier in this chapter (and most of those in Chapter 11) have one thing in common: Every term in the series is positive. So, each of these series is a *positive series*. In contrast, a series that has infinitely many positive and infinitely many negative terms is called an *alternating series.*

Most alternating series flip back and forth between positive and negative terms so that every odd-numbered term is positive and every even-numbered term is negative, or vice versa. This feature adds another spin onto the whole question of convergence and divergence. In this section, I show you what you need to know about alternating series.

Eyeballing two forms of the basic alternating series

The most basic alternating series comes in two forms. In the first form, the odd-numbered terms are negated; in the second, the even-numbered terms are negated.

Without further ado, here's the first form of the basic alternating series:

$$\sum_{n=1}^{\infty}(-1)^n = -1 + 1 - 1 + 1 - \ldots$$

As you can see, in this series the odd terms are all negated. And here's the second form, whose even terms are negated:

$$\sum_{n=1}^{\infty}(-1)^{n-1} = 1 - 1 + 1 - 1 + \ldots$$

Obviously, in whichever form it takes, the basic alternating series is divergent because it never converges on a single sum but instead jumps back and forth between two sums for all eternity. Although the functions that produce these basic alternating series aren't of much interest by themselves, they get interesting when they're multiplied by an infinite series.

Making new series from old ones

You can turn any positive series into an alternating series by multiplying the series by $(-1)^n$ or $(-1)^{n-1}$. For example, here's an old friend, the harmonic series:

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

To negate the odd terms, multiply by $(-1)^n$:

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots$$

To negate the even terms, multiply by $(-1)^{n-1}$:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

Alternating series based on convergent positive series

If you know that a positive series converges, any alternating series based on this series also converges. This simple rule allows you to list a ton of convergent alternating series. For example:

$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n!} = 2 - 2 + \frac{4}{3} - \frac{2}{3} + \frac{4}{15} - \ldots$$

The first series is an alternating version of a geometric series with $r = \frac{1}{2}$. The second is an alternating variation on the familiar p-series with $p = 2$. The third is an alternating series based on a series that I introduce in the earlier section "Rationally solving problems with the ratio test." In each case, the nonalternating version of the series is convergent, so the alternating series is also convergent.

I can show you an easy way to see why this rule works. As an example, I use the first series of the three I just gave you. The value of the positive version of this series is simple to compute by using the formula from Chapter 11:

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2$$

Similarly, if you negate all the terms, the value is just as simple to compute:

$$\sum_{n=0}^{\infty}-\left(\frac{1}{2}\right)^n = -1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \ldots = -2$$

So, if some terms are positive and others are negative, the value of the resulting series must fall someplace between –2 and 2; therefore the series converges.

Using the alternating series test

As I discuss in the previous section, when you know that a positive series is convergent, you can assume that any alternating series based on that series is also convergent. In contrast, some divergent positive series become convergent when transformed into alternating series.

Fortunately, I can give you a simple test to decide whether an alternating series is convergent or divergent.

An alternating series converges if these two conditions are met:

1. **Its defining sequence converges to zero — that is, it passes the *n*th-term test.**
2. **Its terms are non-increasing (ignoring minus signs) — that is, each term is less than or equal to the term before it.**

These conditions are fairly easy to test for, making the alternating series test one of the easiest tests in this chapter. For example, here are three alternating series:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{2} + \ldots$$

$$\sum_{n=2}^{\infty}(-1)^{n}\frac{1}{n\ln n} = \frac{1}{2\ln 2} - \frac{1}{3\ln 3} + \frac{1}{4\ln 4} - \frac{1}{5\ln 5} + \ldots$$

Just by eyeballing them, you can see that each of them meets both criteria of the alternating series test, so they're all convergent. Notice, too, that in each case, the positive version of the same series is divergent. This underscores an important point: When a positive series is convergent, an alternating series based on it is also necessarily convergent; but when a positive series is divergent, an alternating series based on it may be either convergent or divergent.

Technically speaking, the alternating series test is a one-way test: If the series passes the test — that is, if both conditions hold — the series is convergent. However, if the series fails the test — that is, if either condition isn't met — you can draw no conclusion.

In practice, however — and I'm going out on a thin mathematical limb here — I'd say that when a series fails the alternating series test, you have strong circumstantial evidence that the series is divergent.

Why do I say this? First of all, notice that the first condition is the good old-fashioned *n*th-term test. If any series fails this test, you can just chuck it on the divergent pile and get on with the rest of your day.

Second, it's rare when a series — *any series* — meets the first condition but fails to meet the second condition. Sure, it happens, but you really have to hunt around to find a series like that. And even when you find one, the series usually settles down into an ever-decreasing pattern fairly quickly.

For example, take a look at the following alternating series:

$$\sum_{n=2}^{\infty}(-1)^{n-1}\frac{n^2}{2^n}=\frac{1}{2}-1+\frac{9}{8}-1+\frac{25}{32}-\frac{9}{16}+\frac{49}{128}-\dots$$

Clearly, this series passes the first condition of the alternating series test — the *n*th-term test — because the denominator explodes to infinity at a much faster rate than the numerator.

What about the second condition? Well, the first three terms are increasing (disregarding sign), but beyond these terms the series settles into an ever-decreasing pattern. So, you can chop off the first few terms and express the same series in a slightly different way:

$$=\frac{1}{2}-1+\frac{9}{8}-1+\sum_{n=5}^{\infty}(-1)^{n-1}\frac{n^2}{2^n}$$

This version of the series passes the alternating series test with flying colors, so it's convergent. Obviously, adding a few constants to this series doesn't make it divergent, so the original series is also convergent.

So, when you're testing an alternating series, here's what you do:

1. **Test for the first condition — that is, apply the *n*th-term test.**

   If the series fails, it's divergent, so you're done.

2. **If the series passes the *n*th-term test, test for the second condition — that is, see whether its terms *eventually* settle into a constantly-decreasing pattern (ignoring their sign, of course).**

In most cases, you'll find that a series that meets the first condition also meets the second, which means that the series is convergent.

In the rare cases when an alternating series meets the first condition of the alternating series test but doesn't meet the second condition, you can draw no conclusion about whether that series converges or diverges.

These cases really are rare, but I show you one so that you know what to do in case your professor decides to get cute on an exam:

$$-\frac{1}{10}+\frac{1}{9}-\frac{1}{100}+\frac{1}{99}-\frac{1}{1{,}000}+\frac{1}{999}-\dots$$

$$-\frac{1}{10}+\frac{1}{2}-\frac{1}{100}+\frac{1}{3}-\frac{1}{1{,}000}+\frac{1}{4}-\dots$$

Both of these series meet the first criteria of the alternating series test but fail to meet the second, so you can draw no conclusion based upon this test. In fact, the first series is convergent and the second is divergent. Spend a little time studying them and I believe that you'll see why. (***Hint:*** Try to break each series apart into two separate series.)

Understanding absolute and conditional convergence

In the previous two sections, I demonstrate this important fact: When a positive series is convergent, an alternating series based on it is also necessarily convergent; but when a positive series is divergent, an alternating series based on it may be either convergent or divergent.

So, for any alternating series, you have three possibilities:

- An alternating series is convergent, and the positive version of that series is also convergent.
- An alternating series is convergent, but the positive version of that series is divergent.
- An alternating series is divergent, so the positive version of that series must also be divergent.

The existence of three possibilities for alternating series makes a new concept necessary: the distinction between *absolute convergence* and *conditional convergence.*

Table 12-1 tells you when an alternating series is absolutely convergent, conditionally convergent, or divergent.

Table 12-1 Understanding Absolute and Conditional Convergence of Alternating Series

An Alternating Series Is:	*When That Series Is:*	*And Its Related Positive Series Is:*
Absolutely Convergent	Convergent	Convergent
Conditionally Convergent	Convergent	Divergent
Divergent	Divergent	Divergent

Here are a few examples of alternating series that are absolutely convergent:

$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n!} = 2 - 2 + \frac{4}{3} - \frac{2}{3} + \frac{4}{15} - \ldots$$

I pulled these three examples from "Alternating series based on convergent positive series" earlier in this chapter. In each case, the positive version of the series is convergent, so the related alternating series must be convergent as well. Taken together, these two facts mean that each series converges absolutely.

And here are a few examples of alternating series that are conditionally convergent:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{2} + \ldots$$

$$\sum_{n=2}^{\infty}(-1)^{n}\frac{1}{n\ln n} = \frac{1}{2\ln 2} - \frac{1}{3\ln 3} + \frac{1}{4\ln 4} - \frac{1}{5\ln 5} + \ldots$$

I pulled these examples from "Using the alternating series test" earlier in this chapter. In each case, the positive version of the series diverges, but the alternating series converges (by the alternating series test). So each of these series converges conditionally.

Finally, here are a couple of examples of alternating series that are divergent:

$$\sum_{n=1}^{\infty}(-1)^{n-1}n = 1 - 2 + 3 - 4 + \ldots$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n+1} = \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \ldots$$

$$\frac{-1}{10} + \frac{1}{2} - \frac{1}{100} + \frac{1}{3} - \frac{1}{1{,}000} + \frac{1}{4} - \ldots$$

As you can see, the first two series fail the nth-term test, which is also the first condition of the alternating series test, so these two series diverge. As for the third series, it's basically a divergent harmonic series *minus* a convergent geometric series — that is, a divergent series with a finite number subtracted from it — so the entire series diverges.

Testing alternating series

Suppose that somebody (like your professor) hands you an alternating series that you've never seen before and asks you to find out whether it's absolutely convergent, conditionally convergent, or divergent. Here's what you do:

1. **Apply the alternating series test.**

   In most cases, this test tells you whether the alternating series is convergent or divergent:

   a. If it's divergent, you're done! (The alternating series is divergent.)

   b. If it's convergent, the series is either absolutely convergent or conditionally convergent. Proceed to Step 2.

   c. If the alternating series test is inconclusive, you can't rule any option out. Proceed to Step 2.

2. **Rewrite the alternating series as a positive series by:**

   a. Removing $(-1)^n$ or $(-1)^{n-1}$ when you're working with sigma notation.

   b. Changing the minus signs to plus signs when you're working with expanded notation.

3. **Test this positive series for convergence or divergence by using any of the tests in this chapter or Chapter 11:**

   a. If the positive series is convergent, the alternating series is absolutely convergent.

   b. If the positive series is divergent *and* the alternating series is convergent, the alternating series is conditionally convergent.

   c. If the positive series is divergent *but* the alternating series test is inconclusive, the series is either conditionally convergent or divergent, but you still can't tell which.

In most cases, you're not going to get through all these steps and still have a doubt about the series. In the unlikely event that you do find yourself in this position, see whether you can break the alternating series into two separate series — one with positive terms and the other with negative terms — and study these two series for whatever clues you can.

Chapter 13

Dressing up Functions with the Taylor Series

In This Chapter

- Understanding elementary functions
- Seeing power series as polynomials with infinitely many terms
- Expressing functions as a Maclaurin series
- Discovering the Taylor series as a generalization of the Maclaurin series
- Approximating expressions with the Taylor and Maclaurin series

The infinite series known as the Taylor series is one of the most brilliant mathematical achievements that you'll ever come across. It's also quite a lot to get your head around. Although many calculus books tend to throw you in the deep end with the Taylor series, I prefer to take you by the hand and help you wade in slowly.

The Taylor series is a specific form of the power series. In turn, it's helpful to think of a power series as a polynomial with an infinite number of terms. So, in this chapter, I begin with a discussion of polynomials. I contrast polynomials with other elementary functions, pointing out a few reasons why mathematicians like polynomials so much (often, to the exclusion of their families and friends).

Then I move on to power series, showing you how to discover when a power series converges and diverges. I also discuss the interval of convergence for a power series, which is the set of *x* values for which that series converges. After that, I introduce you to the Maclaurin series — a simplified, but powerful, version of the Taylor series.

Finally, the main event: the Taylor series. First, I show you how to use the Taylor series to evaluate other functions; you'll definitely need that for your final exam. I introduce you to the Taylor remainder term, which allows you to find the margin of error when making an approximation. To finish up the chapter, I show you why the Taylor series works, which helps to make sense of the series, but may not be strictly necessary for passing an exam.

Elementary Functions

Elementary functions are those familiar functions that you work with all the time in calculus. They include:

- Addition, subtraction, multiplication, and division
- Powers and roots
- Exponential functions and logarithms (usually, the natural log)
- Trig and inverse trig functions
- All combinations and compositions of these functions

In this section, I discuss some of the difficulties of working with elementary functions. In contrast, I show you why a small subset of elementary functions — the polynomials — is much easier to work with. To finish up, I consider the advantages of expressing elementary functions as polynomials when possible.

Knowing two drawbacks of elementary functions

The set of elementary functions is closed under the operation of differentiation. That is, when you differentiate an elementary function, the result is always another elementary function.

Unfortunately, this set isn't closed under the operation of integration. For example, here's an integral that can't be evaluated as an elementary function:

$$\int e^{x^2}\,dx$$

So, even though the set of elementary functions is large and complex enough to confuse most math students, for you — the calculus guru — it's a rather small pool.

Another problem with elementary functions is that many of them are difficult to evaluate for a given value of x. Even the simple function $\sin x$ isn't so simple to evaluate because (except for 0) every integer input value results in an irrational output for the function. For example, what's the value of sin 3?

Appreciating why polynomials are so friendly

In contrast to other elementary functions, polynomials are just about the friendliest functions around. Here are just a few reasons why:

- Polynomials are easy to integrate (see Chapter 4 to see how to compute the integral of every polynomial).
- Polynomials are easy to evaluate for any value of x.
- Polynomials are infinitely differentiable — that is, you can calculate the value of the first derivative, second derivative, third derivative, and so on, infinitely.

Representing elementary functions as polynomials

In Part II, I show you a set of tricks for computing and integrating elementary functions. Many of these tricks work by taking a function whose integral can't be computed as such and tweaking it into a more friendly form.

For example, using the substitution $u = \sin x$, you can turn the integral on the left into the one on the right:

$$\int \sin^3 x \cos x \, dx = \int u^3 \, du$$

In this case, you're able to turn the product of two trig functions into a polynomial, which is much simpler to work with and easy to integrate.

Representing elementary functions as series

The tactic of expressing complicated functions as polynomials (and other simple functions) motivates much of the study of infinite series.

Although series may seem difficult to work with — and, admittedly, they do pose their own specific set of challenges — they have two great advantages that make them useful for integration:

- First, an infinite series breaks easily into terms. So in most cases, you can use the Sum Rule to break a series into separate terms and evaluate these terms individually.
- Second, series tend to be built from a recognizable pattern. So, if you can figure out how to integrate one term, you can usually generalize this method to integrate every term in the series.

Specifically, power series include many of the features that make polynomials easy to work with. I discuss power series in the next section.

Power Series: Polynomials on Steroids

In Chapter 11, I introduce the geometric series:

$$\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + ax^3 + \ldots$$

I also show you a simple formula to figure out whether the geometric series converges or diverges.

The geometric series is a simplified form of a larger set of series called the power series.

A *power series* is any series of the following form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

Notice how the power series differs from the geometric series:

- In a geometric series, every term has the same coefficient.
- In a power series, the coefficients may be different — usually according to a rule that's specified in the sigma notation.

Here are a few examples of power series:

$$\sum_{n=0}^{\infty} nx^n = x + 2x^2 + 3x^3 + 4x^4 + \ldots$$

$$\sum_{n=0}^{\infty} \frac{1}{2^{n+2}} x^n = \frac{1}{4} + \frac{1}{8} x + \frac{1}{16} x^2 + \frac{1}{32} x^3 + \ldots$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n} = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \ldots$$

You can think of a power series as a polynomial with an infinite number of terms. For this reason, many useful features of polynomials (which I describe earlier in this chapter) carry over to power series.

The most general form of the power series is as follows:

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots$$

This form is for a power series that's centered at a. Notice that when $a = 0$, this form collapses to the simpler version that I introduce earlier in this section. So a power series in this form is centered at 0.

Integrating power series

In Chapter 4, I show you a three-step process for integrating polynomials. Because power series resemble polynomials, they're simple to integrate by using the same basic process:

1. **Use the Sum Rule to integrate the series term by term.**
2. **Use the Constant Multiple Rule to move each coefficient outside its respective integral.**
3. **Use the Power Rule to evaluate each integral.**

For example, take a look at the following integral:

$$\int \sum_{n=0}^{\infty} \frac{1}{2^{n+2}} x^n \, dx$$

At first glance, this integral of a series may look scary. But to give it a chance to show its softer side, I expand the series out as follows:

$$= \int \left(\frac{1}{4} + \frac{1}{8}x + \frac{1}{16}x^2 + \frac{1}{32}x^3 + \ldots \right) dx$$

Now you can apply the three steps for integrating polynomials to evaluate this integral:

1. **Use the Sum Rule to integrate the series term by term:**

$$= \int \frac{1}{4}\, dx + \int \frac{1}{8}x \; dx + \int \frac{1}{16}x^2\, dx + \int \frac{1}{32}x^3\, dx + \ldots$$

2. **Use the Constant Multiple Rule to move each coefficient outside its respective integral:**

$$= \frac{1}{4}\int dx + \frac{1}{8}\int x \; dx + \frac{1}{16}\int x^2\, dx + \frac{1}{32}\int x^3\, dx + \ldots$$

3. **Use the Power Rule to evaluate each integral:**

$$= \frac{1}{4}x + \frac{1}{16}x^2 + \frac{1}{48}x^3 + \frac{1}{128}x^4 + \ldots$$

Notice that this result is another power series, which you can turn back into sigma notation:

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)2^{n+2}} x^{n+1}$$

Understanding the interval of convergence

As with geometric series and *p*-series (which I discuss in Chapter 11), an advantage to power series is that they converge or diverge according to a well-understood pattern.

Unlike these simpler series, however, a power series often converges or diverges based on its x value. This leads to a new concept when dealing with power series: the interval of convergence.

The *interval of convergence* for a power series is the set of x values for which that series converges.

The interval of convergence is never empty

REMEMBER

Every power series converges for some value of x. That is, the interval of convergence for a power series is never the empty set.

Although this fact has useful implications, it's actually pretty much a no-brainer. For example, take a look at the following power series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots$$

When $x = 0$, this series evaluates to $1 + 0 + 0 + 0 + \ldots$, so it obviously converges to 1. Similarly, take a peek at this power series:

$$\sum_{n=0}^{\infty} n(x+5)^n = (x+5) + 2(x+5)^2 + 3(x+5)^3 + 4(x+5)^4 + \ldots$$

This time, when $x = -5$, the series converges to 0, just as trivially as the last example.

Note that in both of these examples, the series converges trivially at $x = a$ for a power series centered at a (see the beginning of "Power Series: Polynomials on Steroids").

Three varieties for the interval of convergence

Three possibilities exist for the interval of convergence of any power series:

- The series converges only when $x = a$.
- The series converges on some interval (open or closed at either end) centered at a.
- The series converges for all real values of x.

For example, suppose that you want to find the interval of convergence for:

$$\sum_{n=0}^{\infty} nx^n = x + 2x^2 + 3x^3 + 4x^4 + \ldots$$

This power series is centered at 0, so it converges when $x = 0$. Using the ratio test (see Chapter 12), you can find out whether it converges for any other values of x. To start out, set up the following limit:

$$\lim_{n \to \infty} \frac{(n+1)\,x^{n+1}}{nx^n}$$

To evaluate this limit, start out by x^n in the numerator and denominator:

$$= \lim_{n \to \infty} \frac{(n+1)\,x}{n}$$

Next, distribute to remove the parentheses in the numerator:

$$= \lim_{n \to \infty} \frac{nx + x}{n}$$

As it stands, this limit is of the form $\frac{\infty}{\infty}$, so apply L'Hospital's Rule (see Chapter 2), differentiating over the variable n:

$$\lim_{n \to \infty} x = x$$

From this result, the ratio test tells you that the series:

- Converges when $-1 < x < 1$
- Diverges when $x < -1$ and $x > 1$
- May converge or diverge when $x = 1$ and $x = -1$

Fortunately, it's easy to see what happens in these two remaining cases. Here's what the series looks like when $x = 1$:

$$\sum_{n=0}^{\infty} n\,(1)^n = 1 + 2 + 3 + 4 + \ldots$$

Clearly, the series diverges. Similarly, here's what it looks like when $x = -1$:

$$\sum_{n=0}^{\infty} n(-1)^n = -1 + 2 - 3 + 4 - \ldots$$

This alternating series swings wildly between negative and positive values, so it also diverges.

As a final example, suppose that you want to find the interval of convergence for the following series:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

As in the last example, this series is centered at 0, so it converges when $x = 0$. The real question is whether it converges for other values of x. Because this is an alternating series, I apply the ratio test to the positive version of it to see whether I can show that it's absolutely convergent:

$$\lim_{n \to \infty} \frac{\dfrac{x^{2(n+1)}}{\big(2(n+1)\big)!}}{\dfrac{x^{2n}}{(2n)!}}$$

First off, I want to simplify this a bit:

$$= \lim_{n \to \infty} \frac{\dfrac{x^{2n+2}}{(2n+2)!}}{\dfrac{x^{2n}}{(2n)!}}$$

$$= \lim_{n \to \infty} \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}}$$

Next, I expand out the exponents and factorials, as I show you in Chapter 12:

$$= \lim_{n \to \infty} \frac{x^{2n} x^2}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{x^{2n}}$$

At this point, a lot of canceling is possible:

$$= \lim_{n \to \infty} \frac{x^2}{(2n+2)(2n+1)} = 0$$

This time, the limit falls between –1 and 1 for all values of x. This result tells you that the series converges absolutely for all values of x, so the alternating series also converges for all values of x.

Expressing Functions as Series

In this section, you begin to explore how to express functions as infinite series. I begin by showing some examples of formulas that express sin x and cos x as series. These examples lead to a more general formula for expressing a wider variety of elementary functions as series.

This formula is the Maclaurin series, a simplified but powerful version of the more general Taylor series, which I introduce later in this chapter.

Expressing sin x as a series

Here's an odd formula that expresses the sine function as an alternating series:

$$\sin x = \sum_{x=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

To make sense of this formula, use expanded notation:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots$$

Notice that this is a power series (which I discuss earlier in this chapter). To get a quick sense of how it works, here's how you can find the value of sin 0 by substituting 0 for x:

$$\sin 0 = 0 - \frac{0^3}{3!} + \frac{0^5}{5!} - \frac{0^7}{7!} + \ldots = 0$$

As you can see, the formula verifies what you already know: sin 0 = 0.

You can use this formula to approximate sin x for any value of x to as many decimal places as you like. For example, look what happens when you substitute 1 for x in the first four terms of the formula:

$$\sin 1 \approx 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5{,}040}$$
$$\approx 0.841468$$

Note that the actual value of sin 1 to six decimal places is 0.841471, so this estimate is correct to five decimal places — not bad!

Table 13-1 shows the value of sin 3 approximated out to six terms. Note that the actual value of sin 3 is approximately 0.14112, so the six-term approximation is correct to three decimal places. Again, not bad, though not quite as good as the estimate for sin 1.

Table 13-1 Approximating the Value of sin 3

# of Terms	*Substitution*	*Approximation*
1	3	3
2	$3-\frac{3^3}{3!}$	–1.5
3	$3-\frac{3^3}{3!}+\frac{3^5}{5!}$	0.525
4	$3-\frac{3^3}{3!}+\frac{3^5}{5!}-\frac{3^7}{7!}$	0.09107
5	$3-\frac{3^3}{3!}+\frac{3^5}{5!}-\frac{3^7}{7!}+\frac{3^9}{9!}$	0.14531
6	$3-\frac{3^3}{3!}+\frac{3^5}{5!}-\frac{3^7}{7!}+\frac{3^9}{9!}-\frac{3^{11}}{11!}$	0.14087

As a final example, Table 13-2 shows the value of sin 10 approximated out to eight terms. The true value of sin 10 is approximately –0.54402, so by any standard this is a poor estimate. Nevertheless, if you continue to generate terms, this estimate continues to get better and better, to any level of precision you like. If you doubt this, notice that after five terms, the approximations are beginning to get closer to the actual value.

Table 13-2 Approximating the Value of sin 10

# of Terms	*Substitution*	*Approximation*
1	10	10
2	$10-\frac{10^3}{3!}$	–156.66667
3	$10-\frac{10^3}{3!}+\frac{10^5}{5!}$	676.66667
4	$10-\frac{10^3}{3!}+\frac{10^5}{5!}-\frac{10^7}{7!}$	–1307.460317
5	$10-\frac{10^3}{3!}+\frac{10^5}{5!}-\frac{10^7}{7!}+\frac{10^9}{9!}$	1448.272
6	$10-\frac{10^3}{3!}+\frac{10^5}{5!}-\frac{10^7}{7!}+\frac{10^9}{9!}-\frac{10^{11}}{11!}$	–1056.938
7	$10-\frac{10^3}{3!}+\frac{10^5}{5!}-\frac{10^7}{7!}+\frac{10^9}{9!}-\frac{10^{11}}{11!}+\frac{10^{13}}{13!}$	548.966
8	$10-\frac{10^3}{3!}+\frac{10^5}{5!}-\frac{10^7}{7!}+\frac{10^9}{9!}-\frac{10^{11}}{11!}+\frac{10^{13}}{13!}-\frac{10^{15}}{15!}$	–215.750

Expressing cos x as a series

In the previous section, I show you a formula that expresses the value of sin x for all values of x as an infinite series. Differentiating both sides of this formula leads to a similar formula for cos x:

$$\frac{d}{dx}\sin x = \frac{d}{dx}x - \frac{d}{dx}\frac{x^3}{3!} + \frac{d}{dx}\frac{x^5}{5!} - \frac{d}{dx}\frac{x^7}{7!} + \ldots$$

Now, evaluate these derivatives:

$$\cos x = 1 - 3\frac{x^2}{3!} + 5\frac{x^4}{5!} - 7\frac{x^6}{7!} + \ldots$$

Finally, simplify the result a bit:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

As you can see, the result is another power series (which I discuss earlier in this chapter). Here's how you write it by using sigma notation:

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

To gain some confidence that this series really works as advertised, note that the substitution $x = 0$ provides the correct equation cos 0 = 1. Furthermore, substituting $x = 1$ into the first four terms gives you the following approximation:

$$\cos 1 \approx 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} = 0.5402\overline{77}$$

This estimate is accurate to four decimal places.

Introducing the Maclaurin Series

In the last two sections, I show you formulas for expressing both sin x and cos x as infinite series. You may begin to suspect that there's some sort of method behind these formulas. Without further ado, here it is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Behold the *Maclaurin series,* a simplified version of the much-heralded Taylor series, which I introduce in the next section.

The notation $f^{(n)}$ means "the nth derivative of f." This should become clearer in the expanded version of the Maclaurin series:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

The Maclaurin series is the template for the two formulas I introduce earlier in this chapter. It allows you to express many other functions as power series by following these steps:

1. **Find the first few derivatives of the function until you recognize a pattern.**
2. **Substitute 0 for x into each of these derivatives.**
3. **Plug these values, term by term, into the formula for the Maclaurin series.**
4. **If possible, express the series in sigma notation.**

For example, suppose that you want to find the Maclaurin series for e^x.

1. **Find the first few derivatives of e^x until you recognize a pattern:**

   $f'(x) = e^x$

   $f''(x) = e^x$

   $f'''(x) = e^x$

   ...

   $f^{(n)}(x) = e^x$

2. **Substitute 0 for x into each of these derivatives.**

   $f'(0) = e^0$

   $f''(0) = e^0$

   $f'''(0) = e^0$

   ...

   $f^{(n)}(x) = e^0$

3. **Plug these values, term by term, into the formula for the Maclaurin series:**

   $$e^x = e^0 + e^0 x + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 + \ldots$$

   $$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

4. **If possible, express the series in sigma notation:**

   $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

To check this formula, use it to estimate e^0 and e^1 by substituting 0 and 1, respectively, into the first six terms:

$$e^0 = 1 + 0 + 0 + 0 + 0 + 0 + \ldots = 1$$

$$e^1 \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = 2.7166\text{(repeating)}$$

This exercise nails e^0 exactly, and approximates e^1 to two decimal places. And, as with the formulas for sin x and cos x that I show you earlier in this chapter, the Maclaurin series for e^x allows you to calculate this function for any value of x to any number of decimal places.

As with the other formulas, however, the Maclaurin series for e^x works best when x is close to 0. As x moves away from 0, you need to calculate more terms to get the same level of precision.

But now, you can begin to see why the Maclaurin series tends to provide better approximations for values close to 0: The number 0 is "hardwired" into the formula as $f(0)$, $f'(0)$, $f''(0)x$, and so forth.

Figure 13-1 illustrates this point. The first graph shows sin x approximated by using the first two terms of the Maclaurin series — that is, as the third-degree polynomial $x - \frac{x^3}{3!}$. The subsequent graph shows an approximation of sin x with four terms.

A tale of three series

It's easy to get confused about the three categories of series that I discuss in this chapter. Here's a helpful way to think about them:

- The *power series* is a subcategory of infinite series.
- The *Taylor series* (named for mathematician Brook Taylor) is a subcategory of power series.
- The *Maclaurin series* (named for mathematician Colin Maclaurin) is a subcategory of Taylor series.

After you have that down, consider that the power series has two basic forms:

- The *specific form*, which is centered at zero, so *a* drops out of the expression.
- The *general form*, which isn't centered at zero, so *a* is part of the expression.

Furthermore, each of the other two series uses one of these two forms of the power series:

- The Maclaurin series uses the specific form, so it's:
  - Less powerful
  - Simpler to work with
- The Taylor series uses the general form, so it's:
  - More powerful
  - Harder to work with

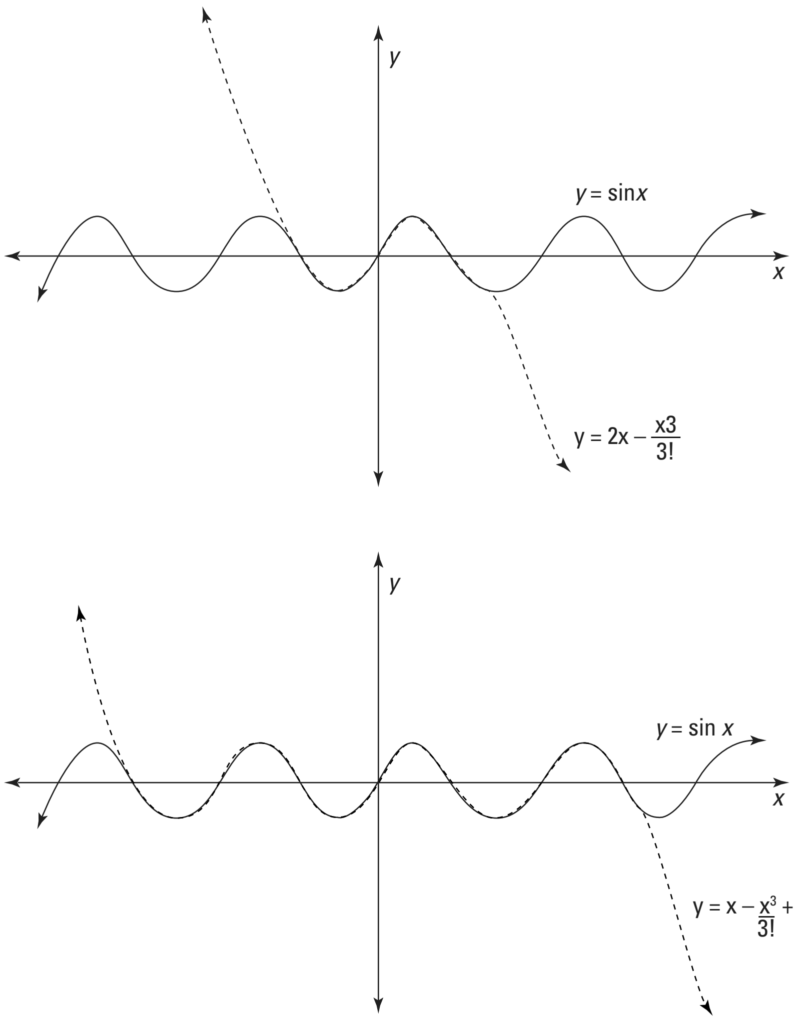

Figure 13-1: Approximating sin *x* by using the Maclaurin series.

As you can see, each successive approximation improves upon the previous one. Furthermore, each equation tends to provide its best approximation when x is close to 0.

Introducing the Taylor Series

Like the Maclaurin series (which I introduce in the previous section), the Taylor series provides a template for representing a wide variety of functions as power series.

In fact, the Taylor series is really a more general version of the Maclaurin series. The advantage of the Maclaurin series is that it's a bit simpler to work with. The advantage to the Taylor series is that you can tailor it to obtain a better approximation of many functions.

Without further ado, here's the Taylor series in all its glory:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

As with the Maclaurin series, the Taylor series uses the notation $f^{(n)}$ to indicate the *n*th derivative. Here's the expanded version of the Taylor series:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

Notice that the Taylor series includes the variable a, which isn't found in the Maclaurin series. Or, more precisely, in the Maclaurin series, $a = 0$, so it drops out of the expression.

The explanation for this variable can be found earlier in this chapter, in "Power Series: Polynomials on Steroids." In that section, I show you two forms of the power series:

- A simpler form centered at 0, which corresponds to the Maclaurin series
- A more general form centered at a, which corresponds to the Taylor series

In the next section, I show you the advantages of working with this extra variable.

Computing with the Taylor series

The presence of the variable a makes the Taylor series more complex to work with than the Maclaurin series. But this variable provides the Taylor series with greater flexibility, as the next example illustrates.

In "Expressing Functions as Series" earlier in this chapter, I attempt to approximate the value of sin 10 with the Maclaurin series. Unfortunately, taking this calculation out to eight terms still results in a poor estimate. This problem occurs because the Maclaurin series always takes a default value of $a = 0$, and 0 isn't close enough to 10.

This time, I use only four terms of the Taylor series to make a much better approximation. The key to this approximation is a shrewd choice for the variable a:

Let $a = 3\pi$

This choice has two advantages: First, this value of a is close to 10 (the value of x), which makes for a better approximation. Second, it's an easy value for calculating sines and cosines, so the computation shouldn't be too difficult.

To start off, substitute 10 for x and 3π for a in the first four terms of the Taylor series:

$$\sin 10 = \sin 3\pi + (\sin' 3\pi)(10 - 3\pi) + \frac{(\sin'' 3\pi)(10 - 3\pi)^2}{2!} + \frac{(\sin''' 3\pi)(10 - 3\pi)^3}{3!}$$

Next, substitute in the first, second, and third derivatives of the sine function and simplify:

$$= \sin 3\pi + (\cos 3\pi)(0.5752) - \frac{(\sin 3\pi)(0.5752)^2}{2!} - \frac{(\cos 3\pi)(0.5752)^3}{3!}$$

The good news is that $\sin 3\pi = 0$, so the first and third terms fall out:

$$= (\cos 3\pi)(0.5752) - \frac{(\cos 3\pi)(0.5752)^3}{3!}$$

At this point, you probably want to grab your calculator:

$$= -1\ (0.5752) - -\frac{1}{6}\ (0.5752)^3$$

$$= -0.5752 + 0.0317 = -0.5434$$

This approximation is correct to two decimal places — quite an improvement over the estimate from the Maclaurin series!

Examining convergent and divergent Taylor series

Earlier in this chapter, I show you how to find the interval of convergence for a power series — that is, the set of x values for which that series converges.

REMEMBER

Because the Taylor series is a form of power series, you shouldn't be surprised that every Taylor series also has an interval of convergence. When this interval is the entire set of real numbers, you can use the series to find the value of $f(x)$ for every real value of x.

However, when the interval of convergence for a Taylor series is bounded — that is, when it diverges for some values of x — you can use it to find the value of $f(x)$ *only* on its interval of convergence.

For example, here are the three important Taylor series that I've introduced so far in this chapter:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

All three of these series converge for all real values of x (you can check this by using the ratio test, as I show you earlier in this chapter), so each equals the value of its respective function.

Now, consider the following function:

$$f(x) = \frac{1}{1-x}$$

I express this function as a Maclaurin series, using the steps that I outline earlier in this chapter in "Expressing Functions as Series":

1. **Find the first few derivatives of $f(x) = \frac{1}{1-x}$ until you recognize a pattern:**

$$f'(x) = \frac{1}{1-x^2}$$

$$f''(x) = \frac{2}{(1-x)^3}$$

$$f'''(x) = \frac{6}{(1-x)^4}$$

...

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$$

2. **Substitute 0 for x into each of these derivatives:**

$f'(0) = 1$

$f''(0) = 2$

$f'''(0) = 6$

...

$f^{(n)}(0) = n!$

3. **Plug these values, term by term, into the formula for the Maclaurin series:**

$$\frac{1}{1-x} = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$
$$= 1 + x + x^2 + x^3 + \ldots$$

4. **If possible, express the series in sigma notation:**

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3$$

To test this formula, I use it to find $f(x)$ when $x = \frac{1}{2}$.

$$f\left(\frac{1}{2}\right) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2$$

You can test the accuracy of this expression by substituting $\frac{1}{2}$ into $\frac{1}{1-x}$:

$$f\left(\frac{1}{2}\right) = \frac{1}{1-\frac{1}{2}} = 2$$

As you can see, the formula produces the correct answer. Now, I try to use it to find $f(x)$ when $x = 5$, noting that the correct answer should be $\frac{1}{1-5} = -\frac{1}{4}$:

$f(5) = 1 + 5 + 25 + 125 + \ldots = \infty$ WRONG!

What happened? This series converges only on the interval (–1, 1), so the formula produces only the value $f(x)$ when x is in this interval. When x is outside this interval, the series diverges, so the formula is invalid.

Expressing functions versus approximating functions

It's important to be crystal clear in your understanding about the difference between two key mathematical practices:

- *Expressing* a function as an infinite series
- *Approximating* a function by using a finite number of terms of series

Both the Taylor series and the Maclaurin series are variations of the power series. You can think of a power series as a polynomial with infinitely many terms. Also, recall that the Maclaurin series is a specific form of the more general Taylor series, arising when the value of a is set to 0.

Every Taylor series (and, therefore, every Maclaurin series) provides the exact value of a function for all values of x where that series converges. That is, for any value of x on its interval of convergence, a Taylor series converges to $f(x)$.

In practice, however, adding up an infinite number of terms simply isn't possible. Nevertheless, you can approximate the value of $f(x)$ by adding up a finite number from the appropriate Taylor series. You do this earlier in the chapter to estimate the value of sin 10 and other expressions.

An expression built from a finite number of terms of a Taylor series is called a *Taylor polynomial,* $T_n(x)$. Like other polynomials, a Taylor polynomial is identified by its degree. For example, here's the fifth-degree Taylor polynomial, $T_5(x)$, that approximates e^x:

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

Generally speaking, a higher-degree polynomial results in a better approximation. And because this polynomial comes from the Maclaurin series, where $a = 0$, it provides a much better estimate for values of e^x when x is near 0. For the value of e^x when x is near 100, however, you get a better estimate by using a Taylor polynomial for e^x with $a = 100$:

$$e^x \approx e^{100} + e^{100}(x-100) + \frac{e^{100}}{2!}(x-100)^2 + \frac{e^{100}}{3!}(x-100)^3 + \frac{e^{100}}{4!}(x-100)^4 + \frac{e^{100}}{5!}(x-100)^5$$

To sum up, remember the following:

- A convergent Taylor series expresses the exact value of a function.
- A Taylor polynomial, $T_n(x)$, from a convergent series approximates the value of a function.

Calculating error bounds for Taylor polynomials

In the previous section, I discuss how a Taylor polynomial approximates the value of a function:

$$f(x) \approx T_n(x)$$

In many cases, it's helpful to measure the accuracy of an approximation. This information is provided by the *Taylor remainder term:*

$$f(x) = T_n(x) + R_n(x)$$

Notice that the addition of the remainder term $R_n(x)$ turns the approximation into an equation. Here's the formula for the remainder term:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad c \text{ between } a \text{ and } x$$

It's important to be clear that this equation is true for one *specific* value of c on the interval between a and x. It does *not* work for just any value of c on that interval.

Ideally, the remainder term gives you the precise difference between the value of a function and the approximation $T_n(x)$. However, because the value of c is uncertain, in practice the remainder term really provides a worst-case scenario for your approximation.

An example should help to make this idea clear. I use the sixth-degree Taylor polynomial for $\cos x$:

$$\cos x \approx T_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

Suppose that I use this polynomial to approximate $\cos 1$:

$$\cos 1 \approx T_6(1) = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720}$$
$$= 0.540\overline{277}$$

How accurate is this approximation likely to be? To find out, utilize the remainder term:

$$\cos 1 = T_6(x) + R_6(x)$$

Adding the associated remainder term changes this approximation into an equation. Here's the formula for the remainder term:

$$R_6(x) = \frac{\cos^{(7)} c}{7!}(x-0)^7$$
$$= \frac{\sin c}{5{,}040}x^7 \qquad c \text{ between } 0 \text{ and } x$$

So, substituting 1 for x gives you:

$$R_6(1) = \frac{\sin c}{5{,}040} \qquad c \text{ between } 0 \text{ and } 1$$

At this point, you're apparently stuck, because you don't know the value of $\sin c$. However, the sin function always produces a number between –1 and 1, so you can narrow down the remainder term as follows:

$$-\frac{1}{5{,}040} \le R_6(1) \le \frac{1}{5{,}040}$$

Note that $\frac{1}{5{,}040} \approx 0.0001984$, so the approximation of cos 1 given by the $T_6(1)$ is accurate to within 0.0001984 in either direction. And, in fact, cos 1 ≈ 0.540302, so:

$$\cos 1 - T_6(1) \approx 0.540302 - 0.540278 = 0.000024$$

As you can see, the approximation is within the error bounds predicted by the remainder term.

Understanding Why the Taylor Series Works

The best way to see why the Taylor series works is to see how it's constructed in the first place. If you read through this chapter until this point, you should be ready to go.

To make sure that you understand every step along the way, however, I construct the Maclaurin series, which is just a tad more straightforward. This construction begins with the key assumption that a function can be expressed as a power series in the first place:

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots$$

The goal now is to express the coefficients on the right side of this equation in terms of the function itself. To do this, I make another relatively safe assumption that 0 is in the domain of $f(x)$. So when $x = 0$, all but the first term of the series equal 0, leaving the following equation:

$$f(0) = c_0$$

This process gives you the value of the coefficient c_0 in terms of the function. Now, differentiate $f(x)$:

$$f'(x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 \ldots$$

At this point, when $x = 0$, all the x terms drop out:

$$f'(0) = c_1$$

So you have another coefficient, c_1, expressed in terms of the function. To continue, differentiate $f'(x)$:

$$f''(x) = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \ldots$$

Again, when $x = 0$, the x terms disappear:

$$f''(0) = 2c_2$$

$$\frac{f''(0)}{2} = c_2$$

By now, you're probably noticing a pattern: You can always get the value of the next coefficient by differentiating the previous equation and substituting 0 for x into the result:

$$f'''(x) = 6c_3 + 24c_4x + 60c_5x^2 + 120c_6x^3 + \ldots$$

$$f'''(0) = 6c_3$$

$$\frac{f'''(0)}{6} = c_3$$

Furthermore, the coefficients also have a pattern:

$$c_0 = f(0)$$

$$c_1 = f'(0)$$

$$c_2 = \frac{f''(0)}{2!}$$

$$c_3 = \frac{f'''(0)}{3!}$$

$$\ldots$$

$$c_n = \frac{f^{(n)}(0)}{n!}$$

Substituting these coefficients into the original equation results in the familiar Maclaurin series from earlier in this chapter:

$$f(x) = f(0) + f'(0)x + \frac{f'(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

To construct the Taylor series, use a similar line of reasoning, starting with the more general form of the power series:

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \ldots$$

In this case, setting $x = a$ gives you the first coefficient:

$$f(a) = c_0$$

Continue to find coefficients by differentiating $f(x)$ and then repeating the process.